Küppers-Lortz instability in a rapidly rotating inviscid magnetoconvection

P. Murali, S. G. Tagare, and P. V. Hanumantha Ravi
Department of Mathematics and Statistics, University of Hyderabad, Hyderabad-500 134, India
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The Küppers-Lortz instability is shown to exist in a rapidly rotating inviscid magnetoconvection. The threshold for the Elsasser number C is approximately 0.3 and the critical angle at which the instability sets in is found to be 59°.

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I. INTRODUCTION

The instability that can arise in a stratified rapidly rotating hydromagnetic system such as the Earth's core and stellar convection zone of rapidly rotating stars have close links with dynamo theory. An account of such instabilities is found in the magnetoconvection (MAC) wave theory of Braginsky [1]. The magnetic field of the Braginsky model is not small and magnetic (Lorentz) force is as potent as the Archimedian (buoyancy) force driving the instability and as the Coriolis force in determining the course of the instabilities, a fact which led Braginsky to christen them as "MAC waves." It is widely believed that planetary and stellar magnetic fields originate from the convection-driven dynamos. The viscous forces have negligible effect on rapidly rotating magnetoconvection. In an inviscid, electrically and thermally conducting fluid when the energy released by the buoyancy force acting on the fluid balances the energy dissipated by joule heating, the transition occurs from the conducting state, with zero fluid velocities and vertical temperature gradient independent of horizontal coordinates, to the convective roll states, with these quantities varying periodically in the horizontal direction.

The generalization of Braginsky's linear theory to include thermal and Ohmic dissipation was carried out by Eltayeb [2]. Rapidly rotating magnetoconvection in a plane layer in the nonlinear regime was studied by Roberts and Stewartson [3]. Roberts and Stewartson considered an inviscid electrically conducting fluid which was kept rotating about an axis parallel to gravity in the presence of an externally impressed magnetic field in the horizontal direction. Sowards [4] further improved Roberts and Stewartson's model by introducing vertical walls parallel to the applied magnetic field. Küppers-Lortz [5] have shown that at the onset of stationary convection, the roll solution could become unstable to perturbations by rolls with a different axis. This led to the onset of complicated time dependence and a possible chaotic behavior right at the threshold for convection when the Taylor number exceeded a critical value. In this paper, we study the Küppers-Lortz instability in the limit of small Rossby number ($R_0 \ll 1$) and for $\kappa/\eta \ll 1$ (κ and η are the coefficients of thermal and magnetic diffusivity, respectively). These assumptions are valid for the core of the solar planets. Here we have considered an inviscid fluid and instead of the Taylor number we have a dimensionless number C, called the Elsasser number, which is a ratio of the Lorentz force to the Coriolis force. In the next section we obtain the threshold value of C for the onset of Küppers-Lortz instability in the limit of small Rossby number and $\kappa/\eta \ll 1$. In Sec. III we summarize our results.

II. THE KÜPPERS-LORTZ INSTABILITY

Let us consider an electrically and thermally conducting inviscid fluid between two horizontal planes with adverse temperature gradients which is kept rotating about z axis with a constant angular velocity $\mathbf{\Omega}=\Omega\hat{\mathbf{e}}_x$. $\mathbf{H}=H_0\hat{\mathbf{e}}_z$ is an externally impressed magnetic field. These planes bound an incompressible, electrically and thermally conducting inviscid fluid of density ρ_0. μ_m is the coefficient of magnetic permeability, η is magnetic diffusivity, g is the acceleration due to gravity. β is the adverse temperature gradient, α and κ are, respectively, coefficients of thermal expansion and thermal diffusivity of the fluid. We use Cartesian system of coordinates whose dimensionless vertical coordinate z and dimensionless horizontal coordinates x,y are scaled on the depth of the layer of fluid, d. The velocity $\mathbf{v}(u,v,w)$, the temperature θ, the time t, the pressure p, and the magnetic field $\mathbf{H}(H_x,H_y,H_z)$ are nondimensionalized by the scales $\kappa/d, \beta d, d^2/\kappa, \rho_0\kappa^2 d^{-2}$, and $\kappa H_0/\eta$.

In the Boussinesq approximation the equations that describe the motion of a rotating and electrically conducting inviscid fluid in an externally impressed magnetic field are

$$\nabla\cdot\mathbf{v}=0\ , \quad \nabla\cdot\mathbf{H}=0\ , \tag{1}$$

$$R_0\left[\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]-C\left[\frac{\partial\mathbf{H}}{\partial z}+R_m(\mathbf{H}\cdot\nabla)\mathbf{H}\right] = -\nabla\left[R_0 p-\frac{1}{8R_0}|\hat{\mathbf{\Omega}}\times\mathbf{v}|^2+CH_z+\frac{CR_m}{2}|\mathbf{H}|^2\right] + R\theta\hat{\mathbf{e}}_z+(\mathbf{v}\times\hat{\mathbf{\Omega}})\ , \tag{2}$$

$$\frac{\partial\theta}{\partial t}+(\mathbf{v}\cdot\nabla)\theta=w+\nabla^2\theta\ , \tag{3}$$

$$R_m\left[\frac{\partial H}{\partial t}-\nabla\times(\mathbf{v}\times\mathbf{H})\right]-\nabla\times(\mathbf{v}\times\hat{\mathbf{e}}_z)=\nabla^2\mathbf{H}\ , \tag{4}$$

where $\hat{\mathbf{\Omega}}$ is a unit vector along the axis of rotation and

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here we assume $\hat{\mathbf{\Omega}}=\hat{\mathbf{e}}_z$. The dimensionless parameters are the Rossby number R_0, the magnetic Prandtl number R_m, the Elsasser number C and R. These dimensionless numbers are defined as follows

$$R_0=\frac{\kappa}{2\Omega d^2}\ ,\quad R_m=\frac{\kappa}{\eta}\ ,\quad C=\frac{\mu_m H_0^2}{8\pi\rho_0\eta\Omega}\ ,$$

and

$$R=\frac{\alpha\beta g d^4}{2\kappa\Omega}\ .$$

The number R is the ratio of forces due to buoyancy to the forces due to rotation and the Elsasser number C is the ratio of Lorentz force to the Coriolis force.

The linear stability analysis of Eqs. (1)–(4) gives the threshold for stationary convection as

$$R_{\rm sc}=\frac{(\pi^2+q_c^2)[C^2\pi^2+(\pi^2+q_c^2)]}{Cq_c^2} \tag{5}$$

where q_c is the wave number of the critical mode and is given by

$$q_c^2=\pi^2\sqrt{1+C^2}\ . \tag{6}$$

The relevant equations for the study of Küppers-Lortz instability when $R_0 \ll 1$ and $R_m \ll 1$ are

$$-CD^2w+R\nabla_h^2\theta-D\omega_z=0\ , \tag{7}$$

$$-CD^2\omega_z+\nabla^2 w=0\ , \tag{8}$$

$$\left[\nabla^2-\frac{\partial}{\partial t}\right]\theta+w=(\mathbf{v}\cdot\nabla)\theta\ , \tag{9}$$

where w and ω_z are the z components of velocity and vorticity ($\boldsymbol{\omega}=\nabla\times\mathbf{v}$). We can write Eqs. (7)–(9) as

$$L_0X+(\Delta R)L_1X=N(X,X)\ , \tag{10}$$

we have dropped the time derivative in Eq. (9) and the vectors $X,N(X,X)$ are defined as following

$$X=\begin{bmatrix} w\\ \omega_z\\ \theta\end{bmatrix}\ ,\quad N(X,X)=\begin{bmatrix}0\\ 0\\ (\mathbf{v}\cdot\nabla)\theta\end{bmatrix}\ . \tag{11}$$

$\Delta R=R-R_{\rm sc}$, L_0, and L_1 are operators defined by

$$L_0=\begin{bmatrix}-CD^2 & D & R\nabla_h^2\\ D\nabla^2 & -CD^2 & 0\\ 1 & 0 & \nabla^2\end{bmatrix}\ , \tag{12}$$

$$L_1=\begin{bmatrix}0 & 0 & \nabla_h^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\ . \tag{13}$$

We assume that a state X represents cylindrical rolls, whose axis is along the y axis. For R slightly greater than $R_{\rm sc}$ (supercritical bifurcation), one can find this state using perturbation theory, by expanding X systematically in powers of ϵ as

$$X=\sum_{n=0}^{\infty}\epsilon^n X_n\ ,\quad X_n=\begin{bmatrix}w_n\\ \omega_{zn}\\ \theta_n\end{bmatrix}\ ,\quad \text{and } R=R_{\rm sc}+\sum_{n=1}^{\infty}R_n\ . \tag{14}$$

Working order by order in ϵ, we obtain the following results.

The Unperturbed State. To the lowest order in ϵ we form Eq. (10) $L_0X_0=0$, which has the following solutions for ideal boundary conditions in w.

$$\begin{aligned} w_0&=\cos q_c x\,\sin\pi z\ ,\\ u_0&=-\frac{\pi}{q_c}\sin q_c x\,\cos\pi z\ ,\\ v_0&=\frac{\pi^2+q_c^2}{C\pi q_c}\sin q_c x\,\cos\pi z\ ,\\ \theta_0&=\frac{1}{\pi^2+q_c^2}\cos q_c x\,\sin\pi z\ ,\\ \omega_{z0}&=\frac{\pi^2+q_c^2}{C\pi}\cos q_c x\,\sin\pi z\ .\end{aligned} \tag{15}$$

To the next order, we have

$$L_0X_2=-R_2L_1X_0+N(X_0,X_1)+N(X_1,X_0)\ . \tag{16}$$

The solvability condition for this order gives,

$$R_2=\frac{R_{\rm sc}}{8(\pi^2+q_c^2)}\ . \tag{17}$$

We observe that $R_2>0$, hence the bifurcation to the convection state is forward for all values of C.

The Perturbed State. Exploring the stability of the basic cylindrical rolls to a set of rolls whose axis makes an angle ψ with x axis is the essence of studying Küppers-Lortz instability. We introduce a perturbation vector $X_1=Ye^{pt}$. We carry out a stability analysis of linearizing in Y. Equations (7)–(9) can be written by linearizing in Y as

$$L_0Y+(\Delta R)L_1Y=N(X,Y)+N(Y,X)+pMY\ , \tag{18}$$

where M is a diagonal matrix given by

$$M=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\ . \tag{19}$$

Equation (18) yields an eigenvalue condition on p, the rate of growth of the perturbation. We expand p and Y as

$$p=\sum_{n=0}^{\infty}\epsilon^n p_n\ , \tag{20a}$$

$$Y=\sum_{n=0}^{\infty}\epsilon^n Y_n\ , \tag{20b}$$

where Y is the column vector of perturbed quantities and we denote the perturbed quantities by tildes.

To the lowest order of ϵ, we have with $p_0=0$

$$L_0 Y_0 = 0 \; . \tag{21}$$

Solution of Eq. (21) is given by

$$\tilde{u}_0 = -\frac{[C\pi^2\cos\psi + (\pi^2+q_c^2)\sin\psi]}{C\pi q_c} \times \sin(k_1 x + k_2 y)\cos\pi z \; ,$$

$$\tilde{v}_0 = -\frac{[(\pi^2+q_c^2)\cos\psi - C\pi^2\sin\psi]}{C\pi q_c} \times \sin(k_1 x + k_2 y)\cos\pi z \; , \tag{22}$$

$$\tilde{w}_0 = \cos(k_1 x + k_2 y)\sin\pi z \; ,$$

$$\tilde{\omega}_{z0} = -\frac{(\pi^2+q_c^2)}{C\pi}\cos(k_1 x + k_2 y)\cos\pi z \; ,$$

$$\tilde{\theta}_0 = -\frac{1}{\pi^2+q_c^2}\cos(k_1 x + k_2 y)\sin\pi z \; ,$$

where k_1 and k_2 are the components of the wave vector $\mathbf{k}$, i.e., $k_1 = q_c\cos\psi$ and $k_2 = q_c\sin\psi$.

To the first order we have

$$L_0 Y_1 = N(X_0, Y_0) + N(Y_0, X_0) + p_1 M Y_0 \; . \tag{23}$$

Solvability condition of the above equation leads to $p_1 = 0$. The solution of (23) with $p_1 = 0$ is found to be

$$\tilde{w}_1 = (A_+\cos\alpha^+ + A_-\cos\alpha^-)\sin 2\pi z \; ,$$

$$\tilde{u}_1 = (D_+\sin\alpha^+ + D_-\sin\alpha^-)\cos 2\pi z \; ,$$

$$\tilde{v}_1 = (C_+\sin\alpha^+ + C_-\sin\alpha^-)\cos 2\pi z \; , \tag{24}$$

$$\tilde{\omega}_{z1} = (B_+\cos\alpha^+ + B_-\cos\alpha^-)\cos 2\pi z \; ,$$

$$\tilde{\theta}_1 = (E_+\cos\alpha^+ + E_-\cos\alpha^-)\sin 2\pi z \; ,$$

where

$$\alpha^{\pm} = (k_1 \pm q_c)x + k_2 y \; ,$$

$$\Delta_{\pm} = 4\pi^2 + 2q_c^2(1\pm\cos\psi) \; ,$$

$$A_{\pm} = -\frac{CR_{sc}\pi q_c^2\sin^2\psi}{(\pi^2+q_c^2)[(4C^2\pi^2+\Delta_{\pm})\Delta_{\pm} - 2R_{sc}q_c^2(1\pm\cos\psi)]} \; ,$$

$$B_{\pm} = \frac{\Delta_{\pm}A_{\pm}}{C\pi} \; , \tag{25}$$

$$C_{\pm} = -\frac{2\pi\sin\psi \mp B_{\pm}(1\pm\cos\psi)}{2q_c(1\pm\cos\psi)} \; ,$$

$$D_{\pm} = \mp\frac{q_c C_{\pm}\sin\psi + 2\pi A_{\pm}}{q_c(1\pm\cos\psi)} \; ,$$

$$E_{\pm} = \frac{2A_{\pm}(\pi^2+q_c^2) - \pi(1\mp\cos\psi)}{2(\pi^2+q_c^2)\Delta_{\pm}} \; .$$

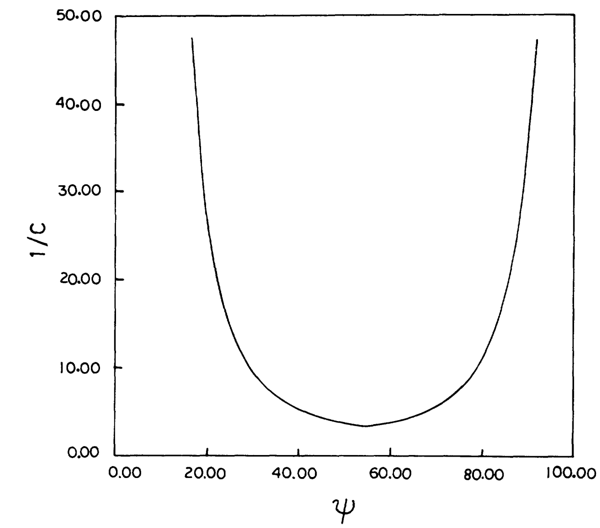

FIG. 1. Plot of ψ versus $1/C$, where C is the ratio of Lorentz force to Coriolis force.

To the next order of ϵ, we obtain

$$L_0 Y_2 + R_2 L_1 Y_0 = N(X_0, Y_1) + N(X_1, Y_0) + N(Y_0, X_1) + N(Y_1, X_0) + p_2 M Y_0 \; . \tag{26}$$

Solvability criterion of above equation is given by

$$p_2 = \frac{(\pi^2+q_c^2)^2}{4C\pi}[E_+ + E_-] + \frac{\pi(\pi^2+q_c^2)}{2} - \frac{\pi(\pi^2+q_c^2)}{4}[E_+(1+\cos\psi) + E_-(1-\cos\psi)] - \frac{q_c}{4}[D_+ - D_-] - \frac{\pi}{4}[A_+ + A_-] \; . \tag{27}$$

The condition for Küppers-Lortz instability is thus given by $p_2 = 0$.

III. RESULTS

We have derived the condition for Küppers-Lortz (KL) instability for Rayleigh-Bènard convection in the presence of vertical magnetic field in an inviscid fluid. We have also calculated numerically the threshold for the Elsasser number C (Fig. 1) and is found approximately to be 0.3. We find that the angle ψ is weakly dependent on the magnetic field. An important observation is that the absence of viscosity does not stabilize the system against KL instability.

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[1] S. I. Braginsky, Geomagn. Aeron. 7, 851 (1967).
[2] I. A. Eltayeb, Proc. R. Soc. London Ser. A **326**, 229 (1972).
[3] P. H. Roberts and K. Stewartson, Philos. Trans. R. Soc. London Ser. A **277**, 287 (1974).
[4] A. M. Sowards, Geophys. Astrophys. Fluid Dyn. **35**, 329 (1986).
[5] G. Küppers and D. Lortz, J. Fluid Mech. **35**, 609 (1969).